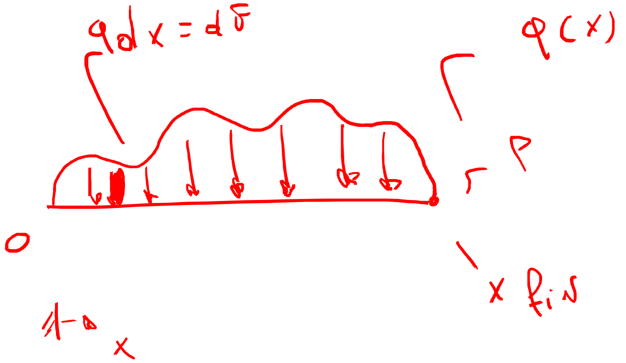


$$F = \int_0^{x_{fin}} q(x)\,dx$$

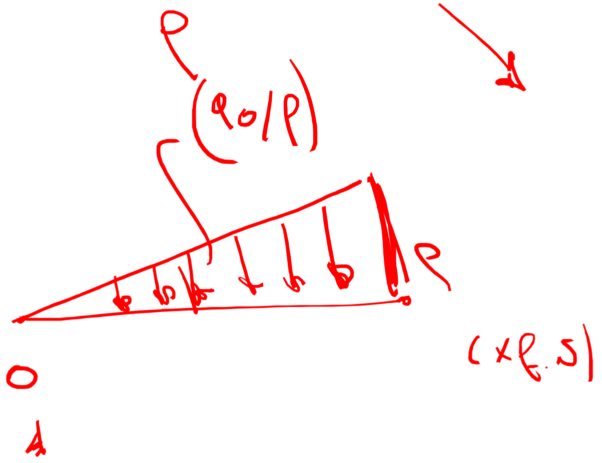
$$M = \int_0^{x_{fin}} q(x)\,x\,dx$$

$$\cancel{q = q_0}$$

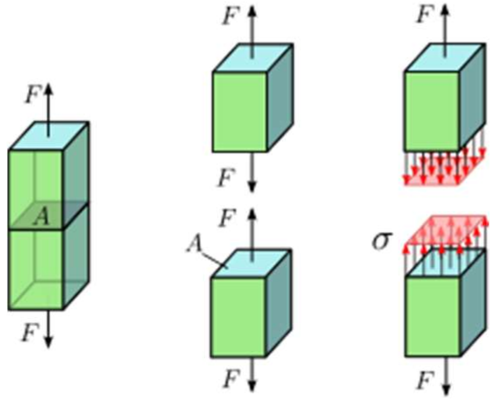
$$q = \frac{x}{\ell} q_0$$

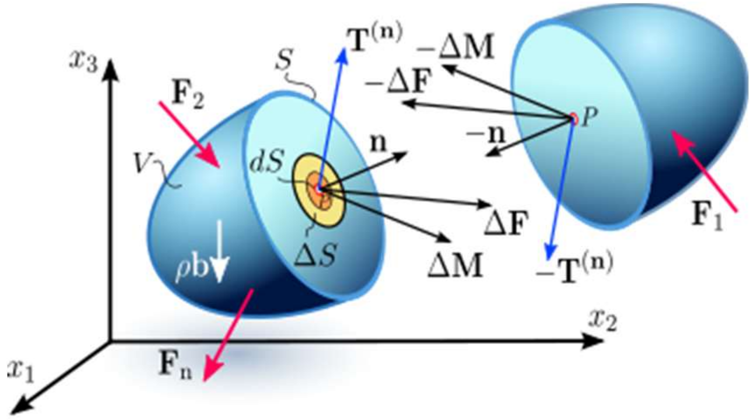
$$F = \int_0^{x_{fin}} \frac{q_0}{\ell}\,x\,dx = \frac{q_0}{\ell}\,\frac{x_{fin}^2}{2}$$

$$M = \int_0^{x_{fin}} \frac{q_0}{\ell}\,x^2\,dx = \frac{q_0}{\ell}\,\frac{x_{fin}^3}{3}$$



STRESS





$$\boldsymbol{t}^{(n)} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{F}}{\Delta S}$$

$$\boldsymbol{t}^{(n)} = \boldsymbol{\sigma}_n + \boldsymbol{\tau}_{\boldsymbol{n}}$$

normal component

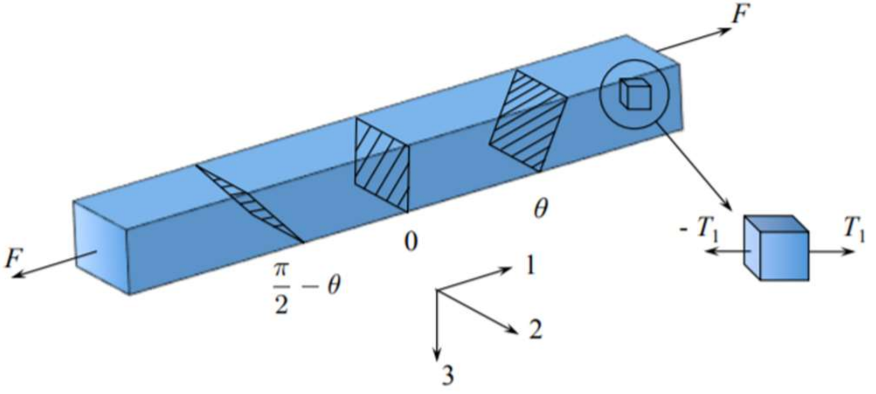
tangential component

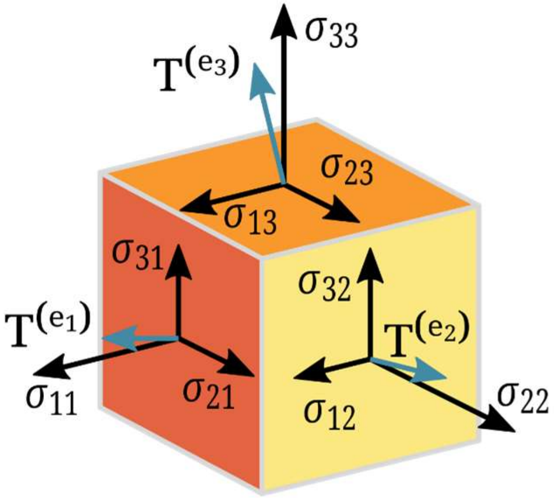
$$\boxed{\boldsymbol{t}^{(n)} = \boldsymbol{\sigma}\,\boldsymbol{n}}$$

Stress Tensor

$$t_i^{(n)} = \sigma_{ij}\, n_j$$

STRESS





In axis components, **normal**

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Off-axis components, **tangential**

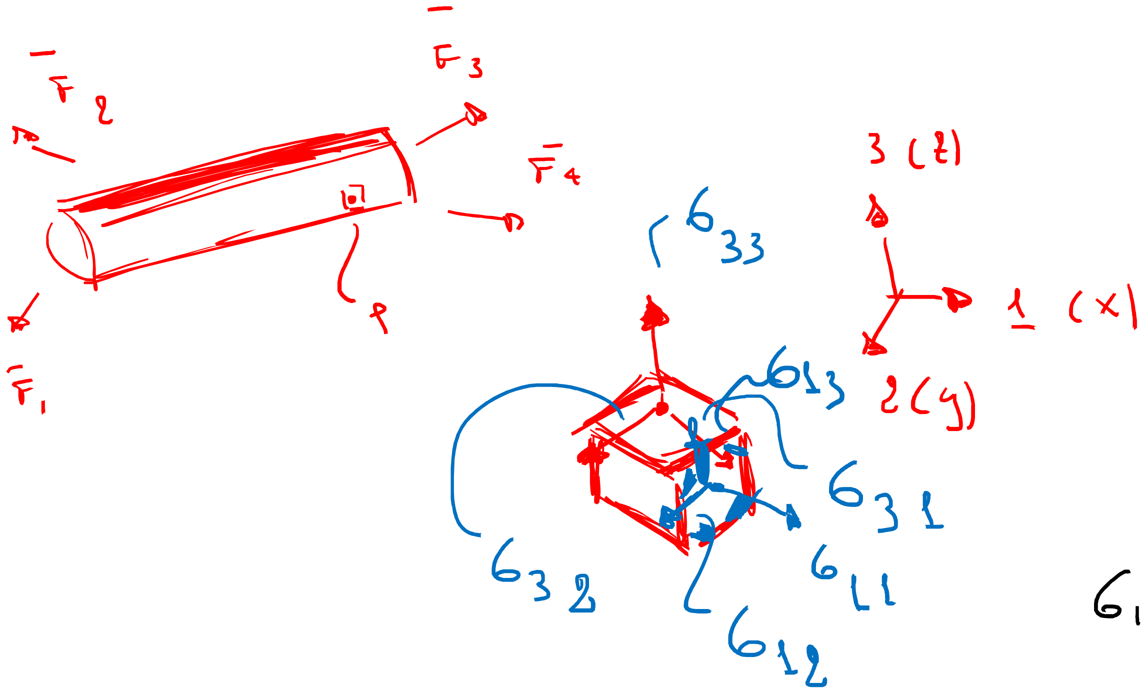
1. The tensor is symmetric
2. It can be diagonalizable
3. For any surface, it is always possible find a system of reference respet to which the stress has only normal components – these are the principal components

$\bar{t}^{(\bar{n})} = \underline{\sigma}\,\bar{n}$ → versore della sezione

Tensore delle tensioni:

Tensione che si scarica in un punto, lungo la normale $\bar{n}$

$$\begin{array}{cccc} 1 & 11 & 12 & 13 \\ 2 & 21 & 22 & 23 \\ 3 & 31 & 32 & 33 \end{array}$$



$$\sigma_{ij}: \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

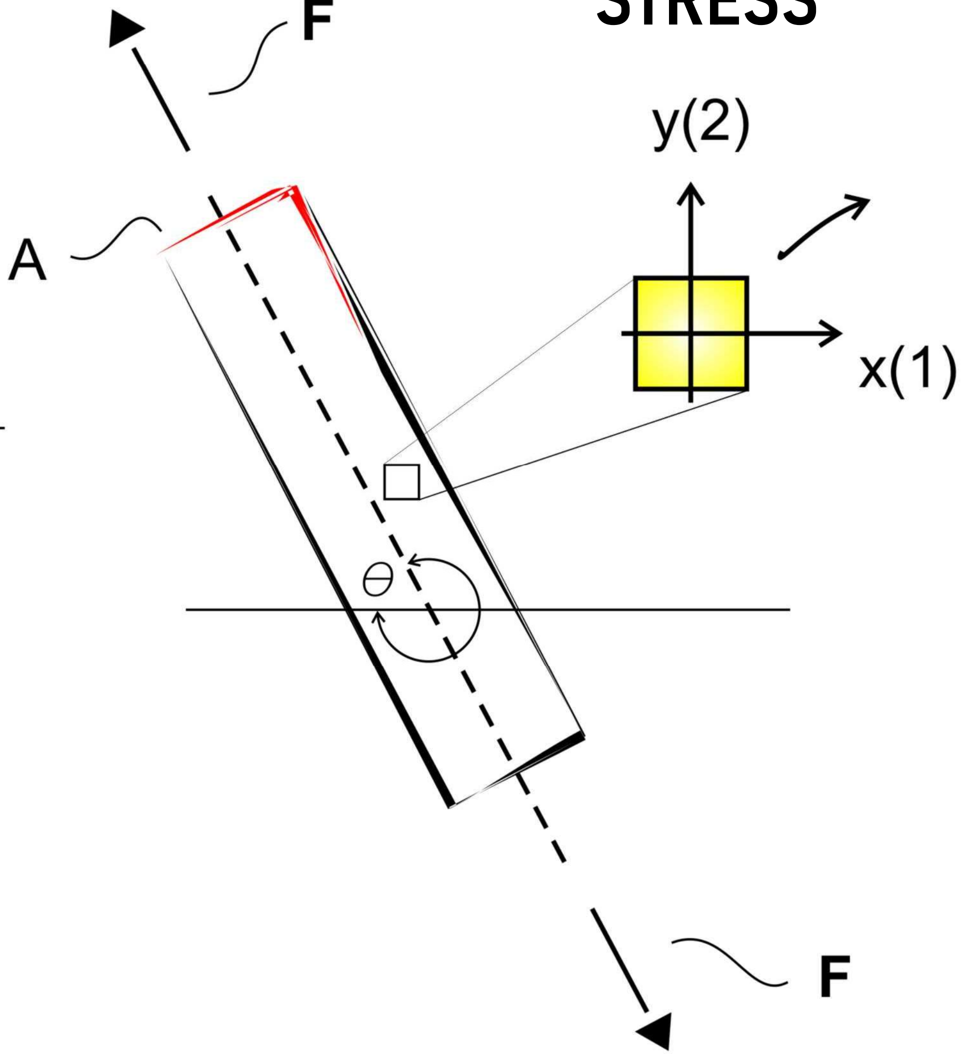
↓

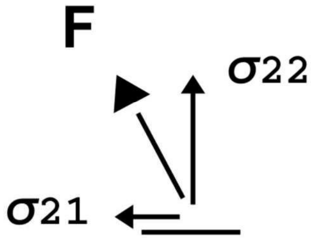
$$\sigma_{ij} = \sigma_{ji}$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

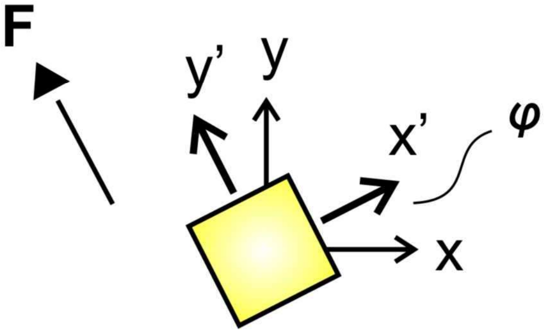
STRESS

$$\sigma_\circ \rightarrow \frac{F}{A}$$





$$\begin{array}{c|c} \sigma_{21} & \sigma_\circ\cos(\theta) \\ \hline \sigma_{22} & \sigma_\circ\sin(\theta) \end{array} \quad \textbf{x-y}$$



$$\textbf{x'-y'} \quad \begin{array}{c|c} \sigma_{21} & 0 \\ \hline \sigma_{22} & \sigma_\circ \end{array} \quad \begin{array}{l} \sigma_{II} \\ \\ \sigma_{I} \end{array}$$

STRESS

$$\boldsymbol{t}^{(n)} = \boldsymbol{\sigma}\,\boldsymbol{n}$$

$$t_i = \sigma_{ij}\,n_j$$

The principal stresses represent the maximum and minimum possible values of the stress tensor expressed in any frame or reference

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Stress tensor

$$\boldsymbol{\sigma}(\boldsymbol{p}) = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix}$$

Stress tensor written in terms of principal stress components

STRESS

$$\boldsymbol{\sigma}(\boldsymbol{p})\,\boldsymbol{n} = \sigma\,\boldsymbol{n}$$

$$(\boldsymbol{\sigma}(\boldsymbol{p}) - \lambda\boldsymbol{I})\boldsymbol{n} = 0$$

$$|\boldsymbol{\sigma}(\boldsymbol{p}) - \lambda\boldsymbol{I}| = 0$$

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

If $\boldsymbol{\sigma}(\boldsymbol{p})$ is such that its tangential, off-axis components are zero, then $\boldsymbol{\sigma}(\boldsymbol{p})$ applied to any surface vector $\boldsymbol{n}$ will result in a stress vector that has the same direction of the original $\boldsymbol{n}$

This is an ***eigenvalue*** problem

In linear algebra, an eigenvector or characteristic vector of a linear transformation is a nonzero vector that changes by a scalar factor when that linear transformation is applied to it. The corresponding eigenvalue is the factor by which the eigenvector is scaled.

STRESS

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} \\ \sigma_{12} & \sigma_{22}-\lambda \end{vmatrix} = (\sigma_{11}-\lambda)(\sigma_{22}-\lambda) - \sigma_{12}^2 = 0$$ In 2d coordinates

$$\sigma_{11}\sigma_{22} - \lambda\,\sigma_{11} - \lambda\,\sigma_{22} + \lambda^2 - \sigma_{12}^2 = 0$$

$$\underbrace{\lambda^2}_{a} - \underbrace{(\sigma_{11} + \sigma_{22})}_{b}\lambda + \underbrace{\sigma_{11}\sigma_{22} - \sigma_{12}^2}_{c} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

STRESS

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

In 3d coordinates

$$\lambda^3 - I_1\lambda^2 + I_2\,\lambda\ - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

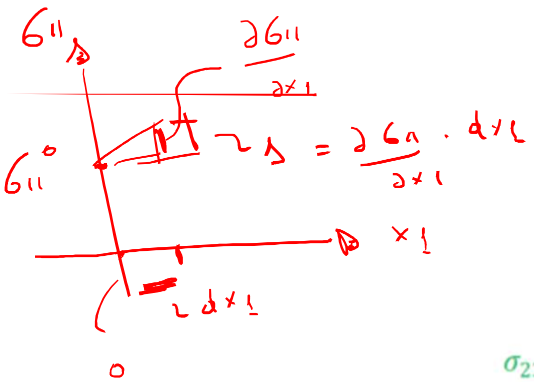
$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$I_3 = |\boldsymbol{\sigma}|$$

STRESS

Find the principal stresses and principal axes for:

$$\begin{pmatrix} 80 & 30 \\ 30 & 40 \end{pmatrix}$$



STRESS

x_1: σ_{11}

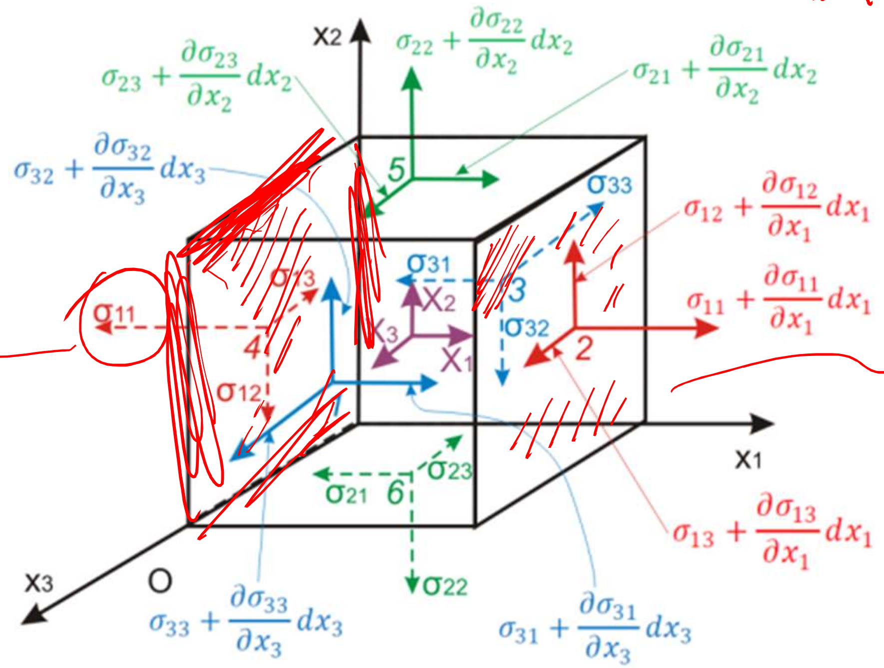
$\sigma_{11} + \frac{\partial\sigma_{11}}{\partial x_1} dx_1$

$\Rightarrow F_{TOT}^{x_1} = \left(\cancel{\sigma_{11}} + \frac{\partial\sigma_{11}}{\partial x_1} dx_1\right) dx_2 dx_3$

$= - \sigma_{11}\, dx_2\, dx_3$

$F_{TOT}^{x_1} = \frac{\partial\sigma_{11}}{\partial x_1} dV$

σ_{11}



$\sigma_{11} + \frac{\partial\sigma_{11}}{\partial x_1} dx_1$

$F_{TOT}^{x_1} = 0$

$$\boxed{\frac{\partial\sigma_{11}}{\partial x_1} dV = 0}$$

$$\sigma_{ij,j} + f_j = 0$$

← For the **equilibrium**, it has to be that

$\sigma_{11,1} = 0 \Leftarrow \frac{\partial\sigma_{11}}{\partial x_1} = 0$

$$\sigma_{ij,j} = \frac{\partial \sigma_i}{\partial x_j}$$

$$\varepsilon_{12} = \frac{\Delta l}{l}$$

$l_1 = 10$ cm

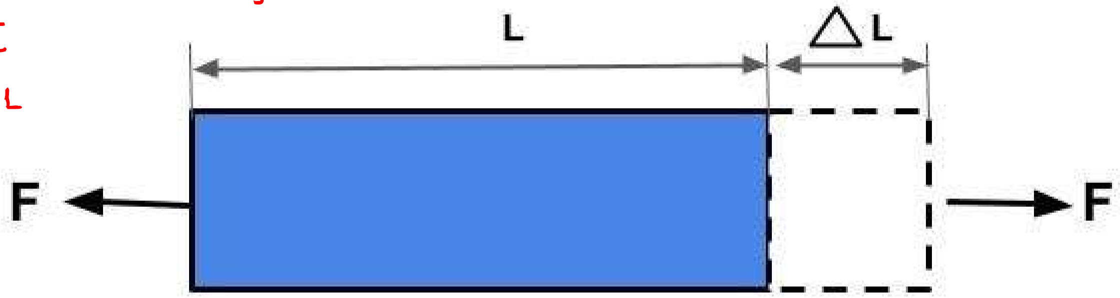
$l_2 = 20$ cm

18 cm

15 cm

$$\varepsilon_{12} = \ln \frac{l_2}{l_1}$$

STRAIN



$$\varepsilon = \text{Strain} = \frac{\text{Change in Length}}{\text{Original Length}} = \frac{\triangle L}{L}$$

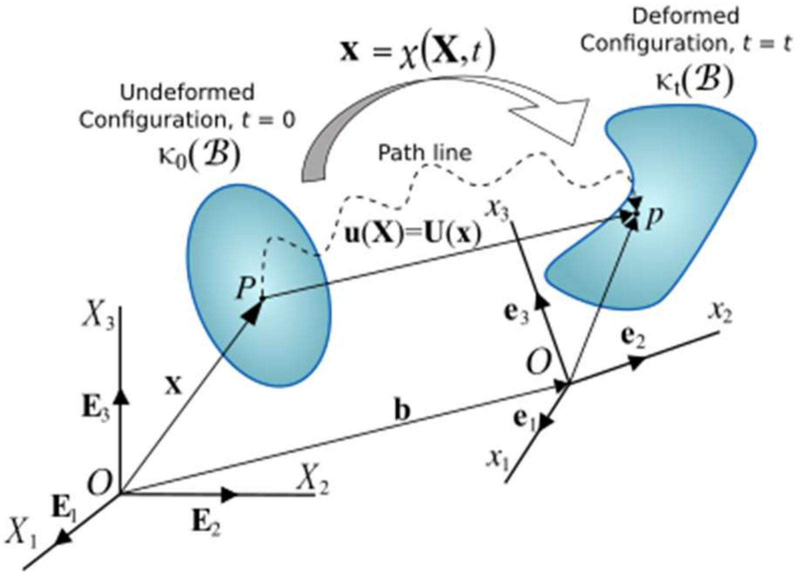
$$\epsilon = \frac{\Delta l}{l_o}$$

$$d\epsilon = \frac{dl}{l} \longrightarrow \int d\epsilon = \int \frac{dl}{l} \longrightarrow \int d\epsilon = \int \frac{dl}{l} \longrightarrow \epsilon = ln\left(\frac{l_2}{l_1}\right)$$

STRAIN

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}$$

The strain is a tensor



Compatibility

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

LINEAR ELASTIC MATERIAL

$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$

Elastic constitutive matrix

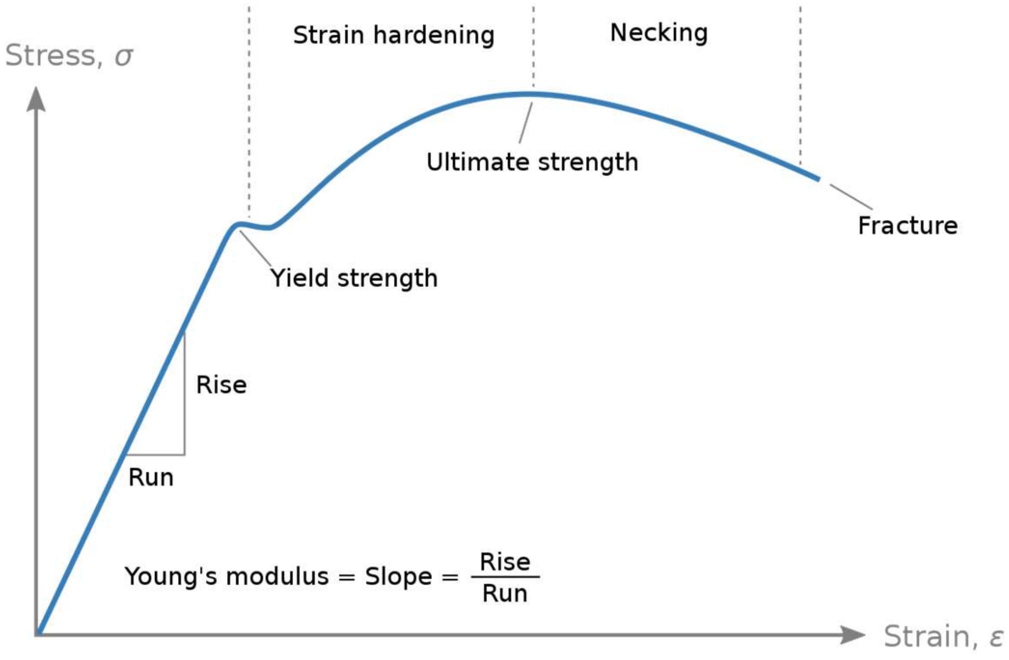
$\boldsymbol{C} = \boldsymbol{C}(E, \nu)$ For an elastic, istropic material

Poisson’s ratio

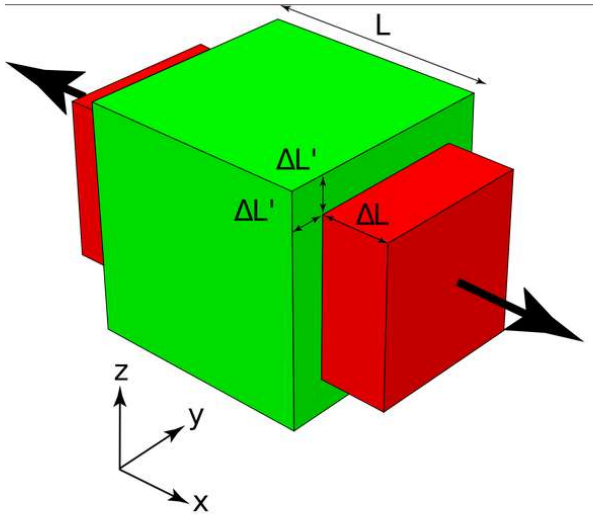
Young’s modulus

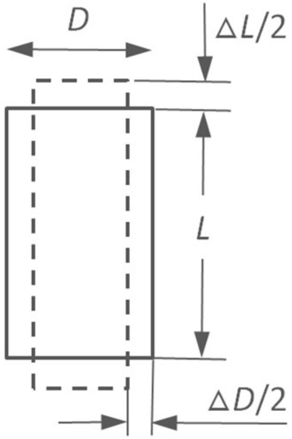
$$\boldsymbol{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

Young’s modulus

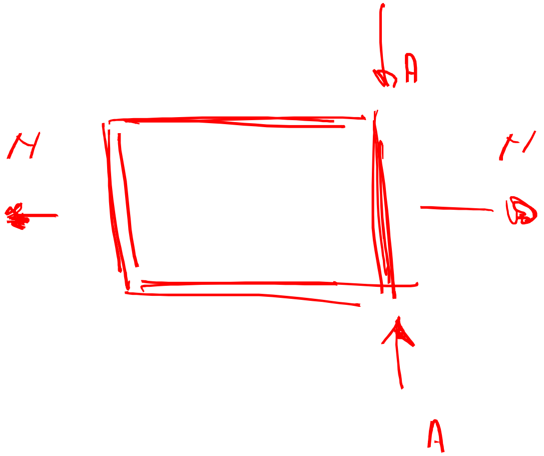


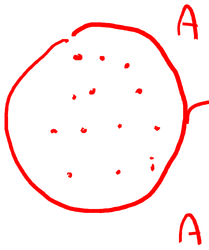
Poisson's ratio





$$\nu = \frac{lateral\ strain}{axial\ strain}$$





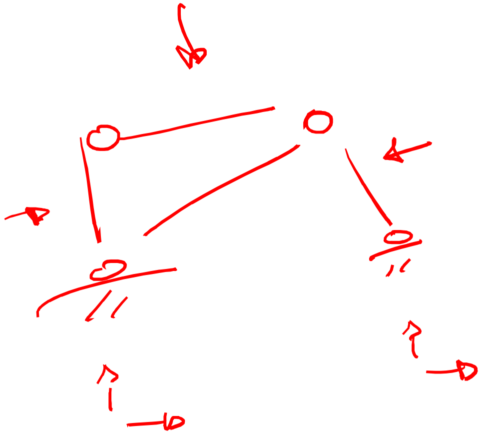
$$\sigma_M \xrightarrow{f(M)} \sigma_M < \sigma_y$$

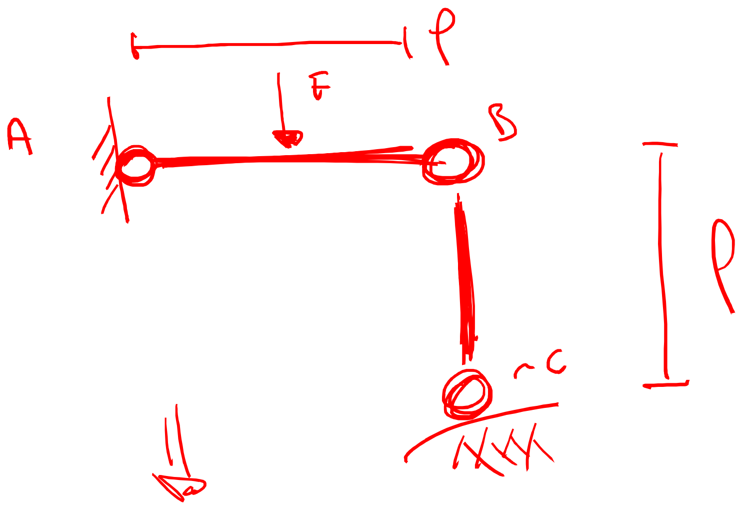
Tensioni normali

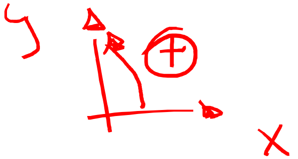
$< \sigma_u$

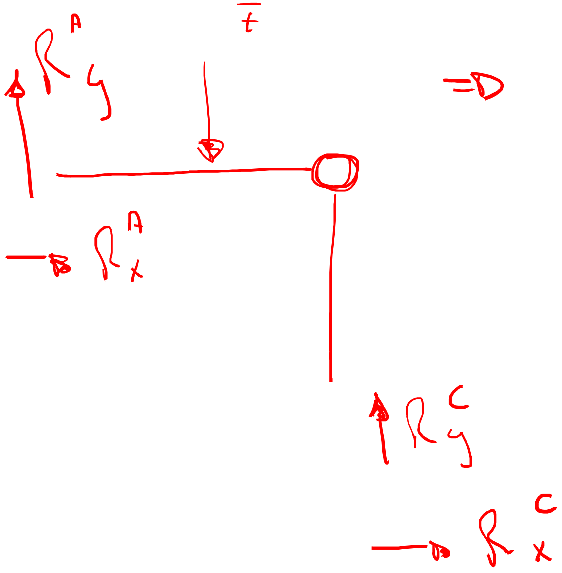
$\sigma_M \to M$

$\hookrightarrow \frac{A}{I}$

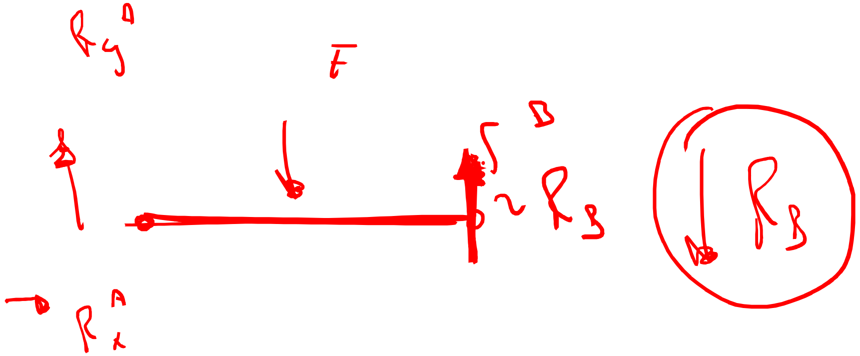








$\Rightarrow$

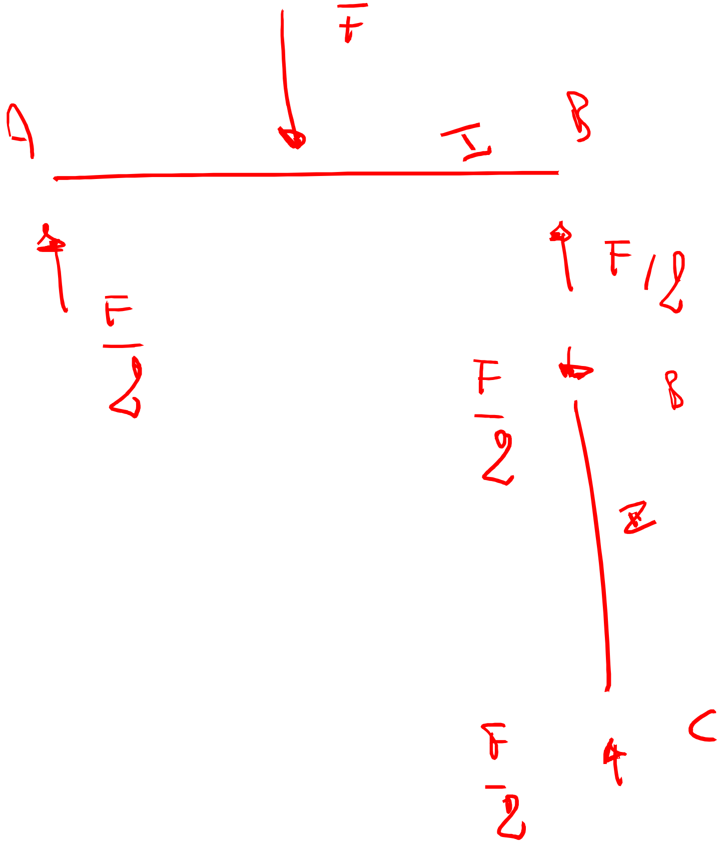


$x: R_A^x = 0$

$y: R_y^A + R_B = F$

$O|_A: -F\frac{l}{2} + R_B \cdot l = 0 \Rightarrow$ $R_B = \frac{F}{2}$

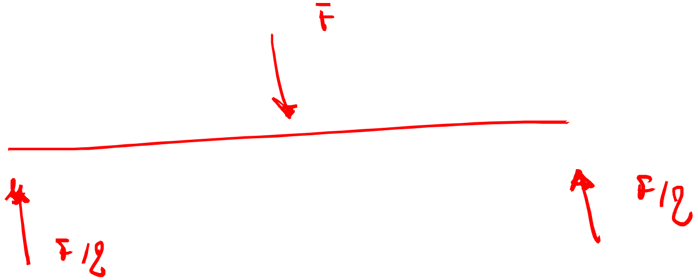
$R_y^A = \frac{F}{2}$

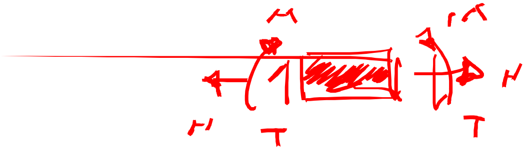


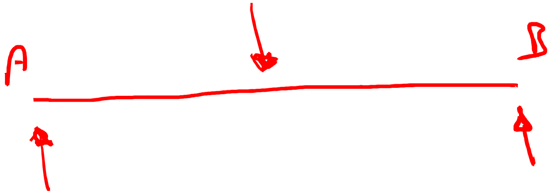
[M]

[T]

[N]



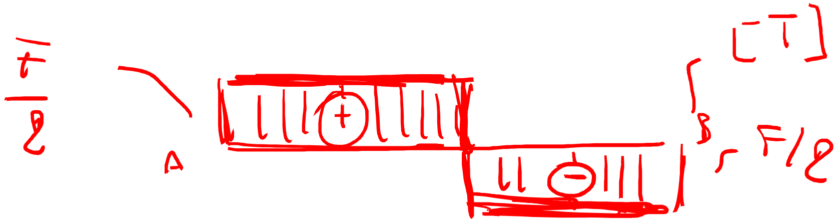




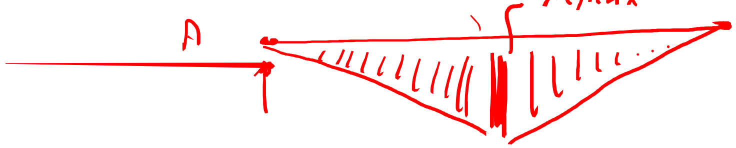
• N : $N = 0$

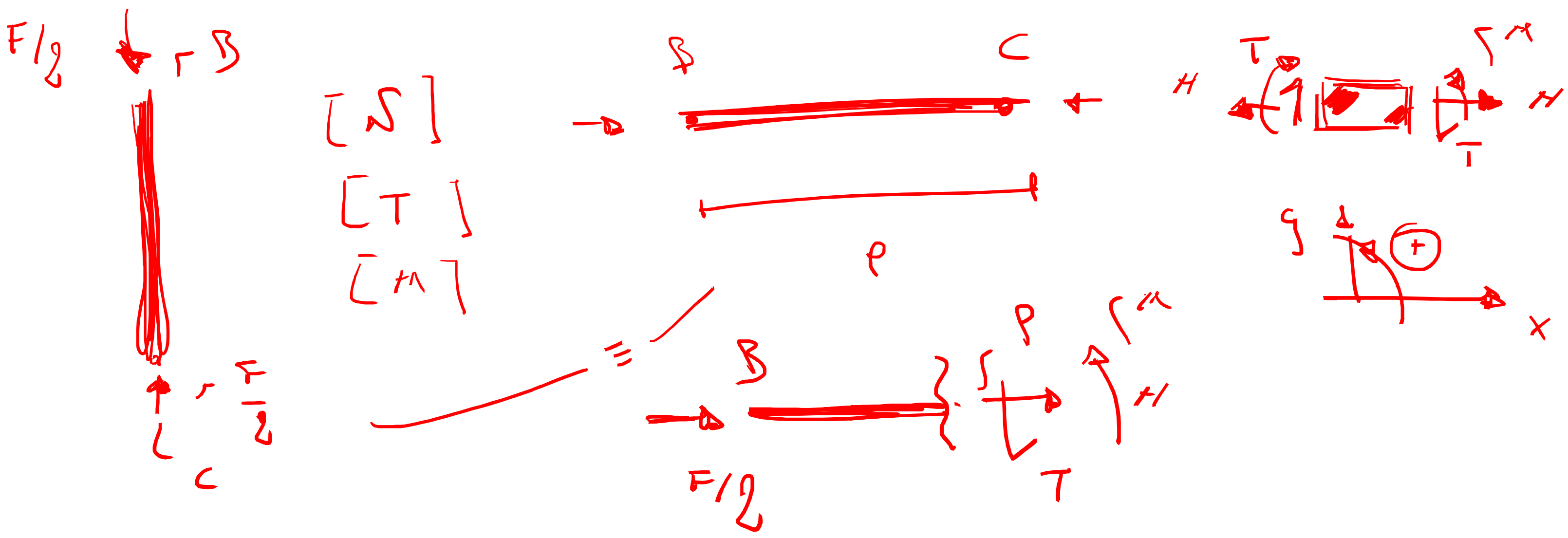
• T : $\frac{F}{2} = T$

• $M\Big|_B$: $-\frac{F}{2}x + M = 0 \Rightarrow M = \frac{F}{2}x$



$M_{max} = F/2 \cdot l/2$



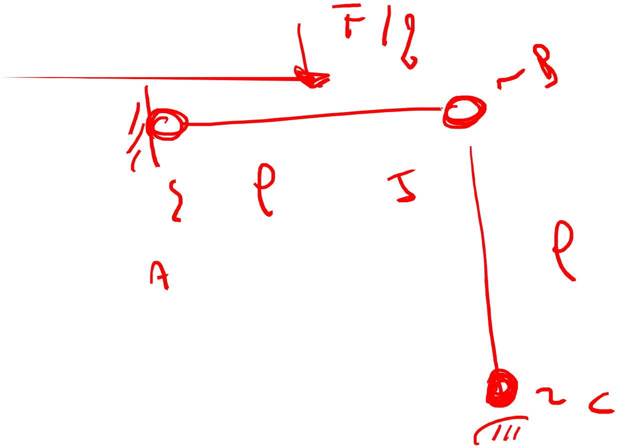


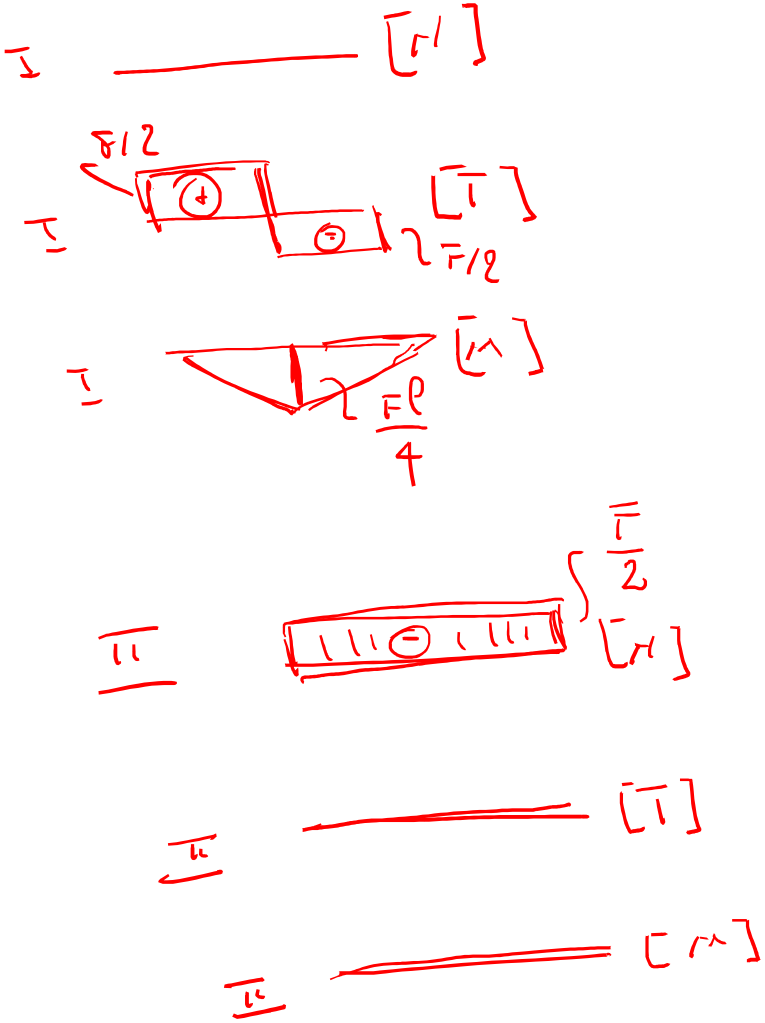
$\Rightarrow N : \frac{F}{2} = S \Rightarrow N = -\frac{F}{2}$ →

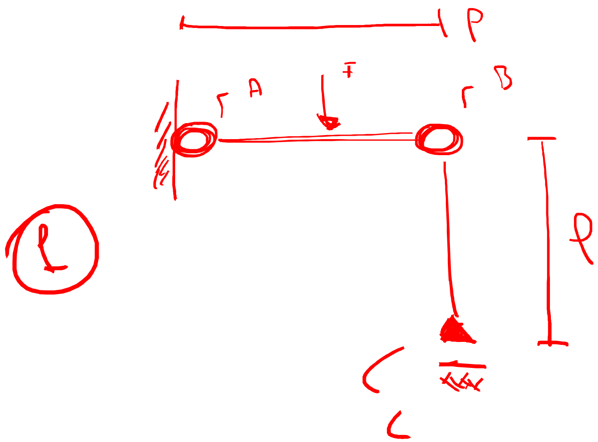
Forze e le tensioni: (stati tensionali) normali negativi sono sempre compressivi.

$\Rightarrow T : T = 0$ ■

$\Rightarrow M : M = 0$ ■


F/2
B
A
l
I
l
C


I
[N]
F/2
I
[T]
F/2
I
[M]
Fl/4
II
F/2
[M]
II
[T]
II
[M]


ℓ
A
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B
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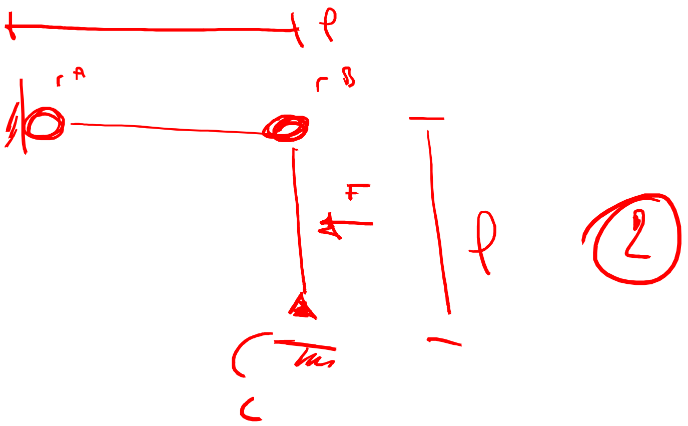
C
①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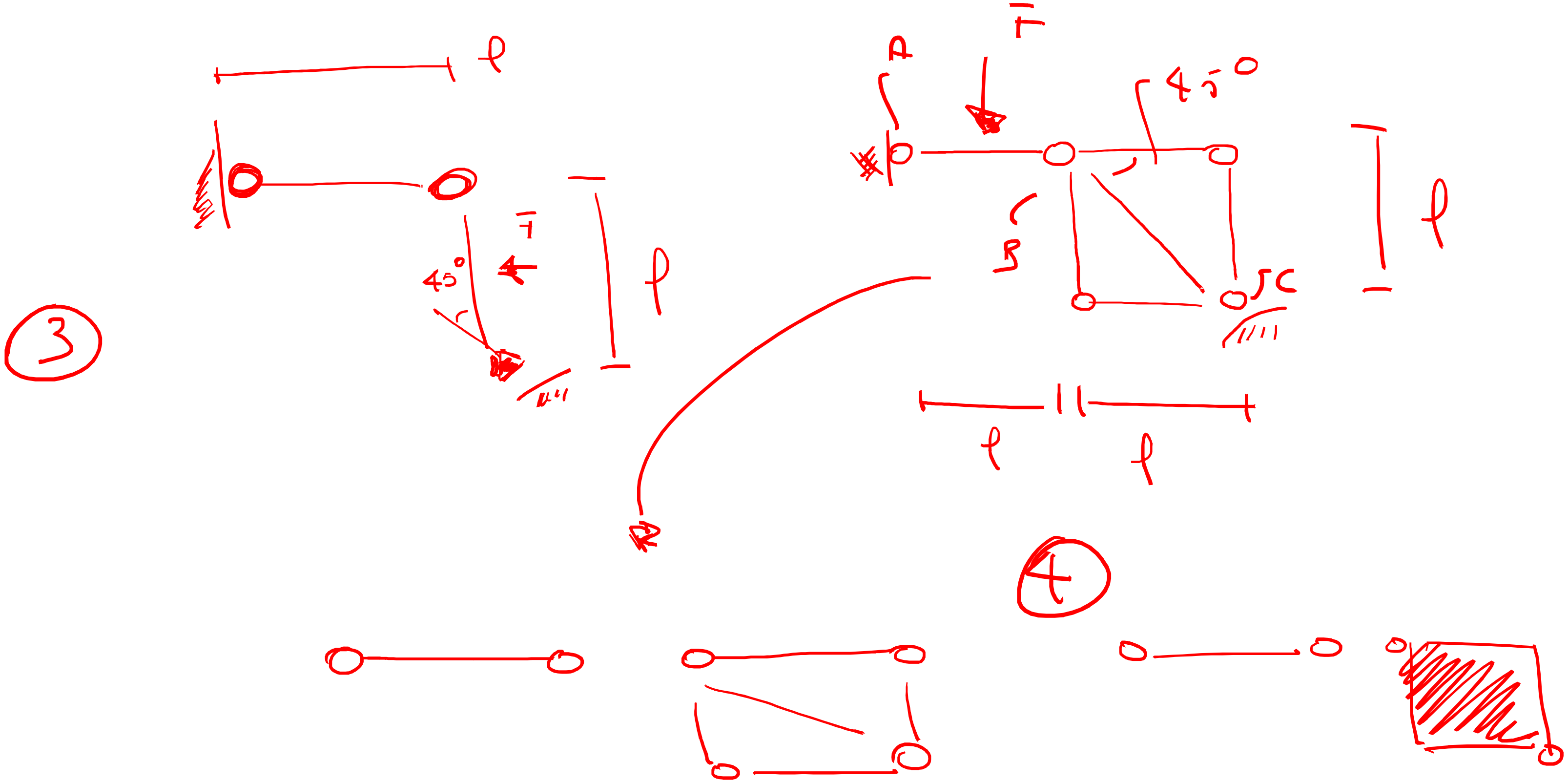
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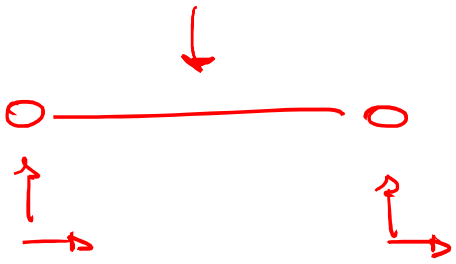
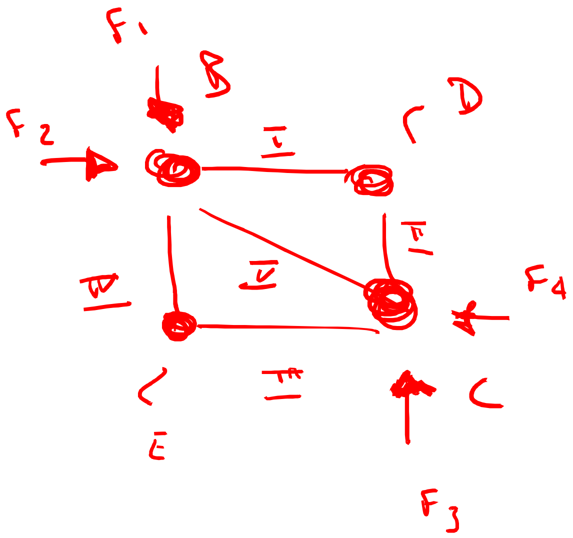
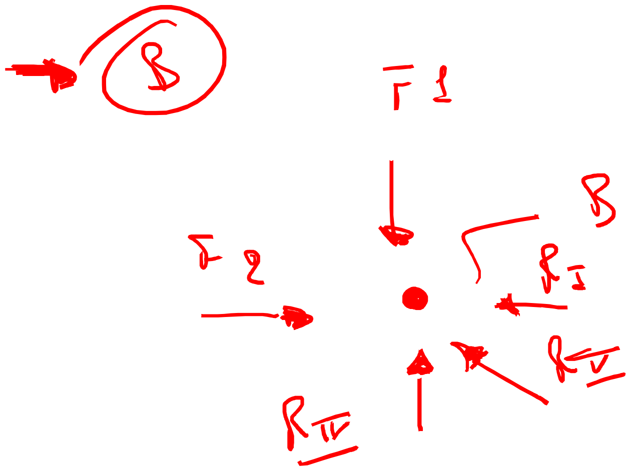
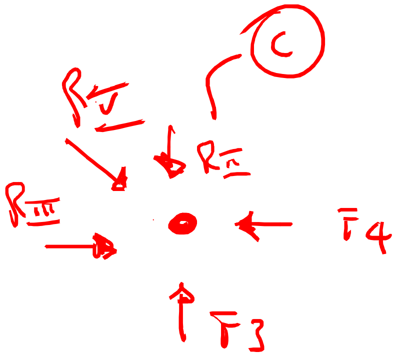


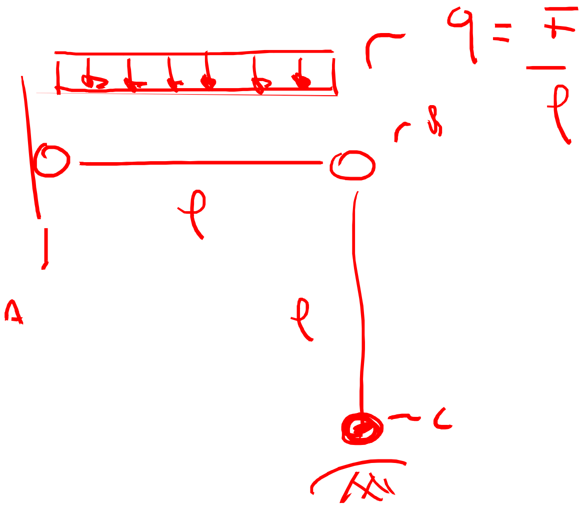

F1
B
D
F2
I
II
IV
V
F4
III
C
E
F3


B
F1
F2
R_{I}
R_{V}
R_{IV}

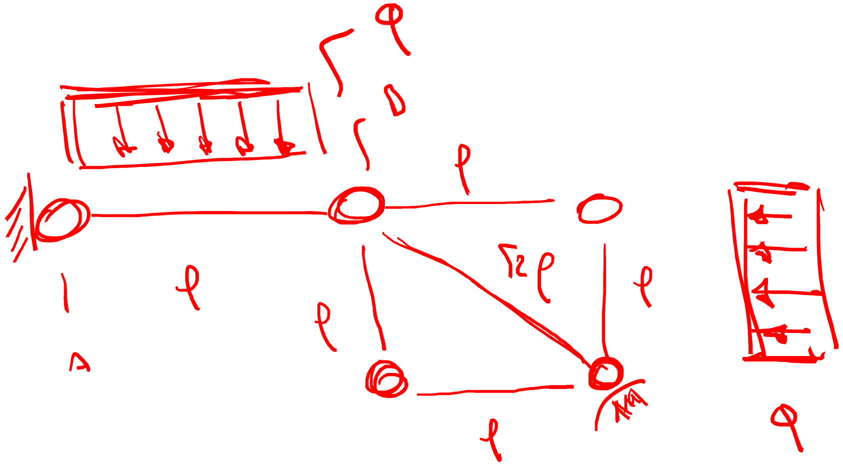

D
R_{I}
R_{II}


E
R_{IV}
R_{III}


C
R_{V}
R_{II}
R_{III}
F4
F3



⑤

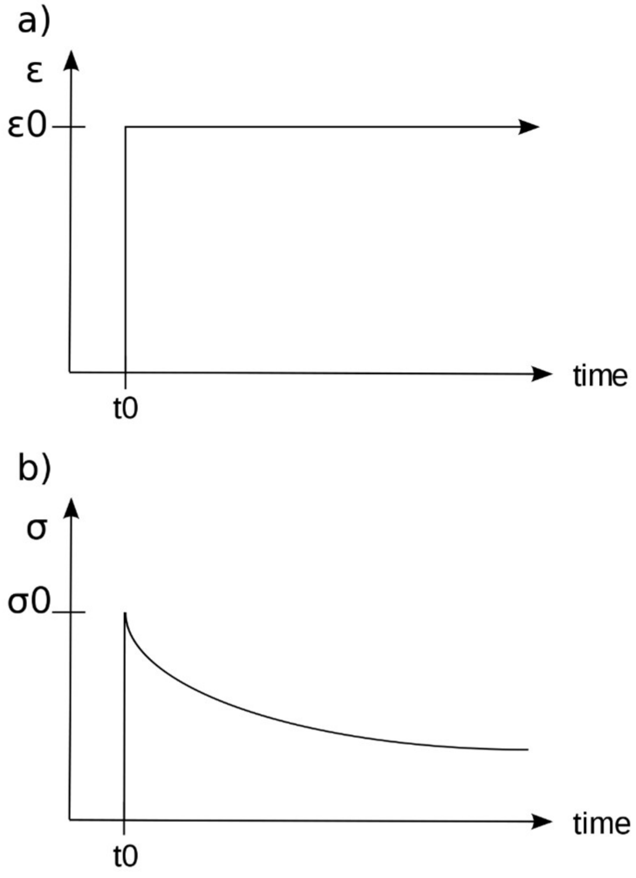


⑥

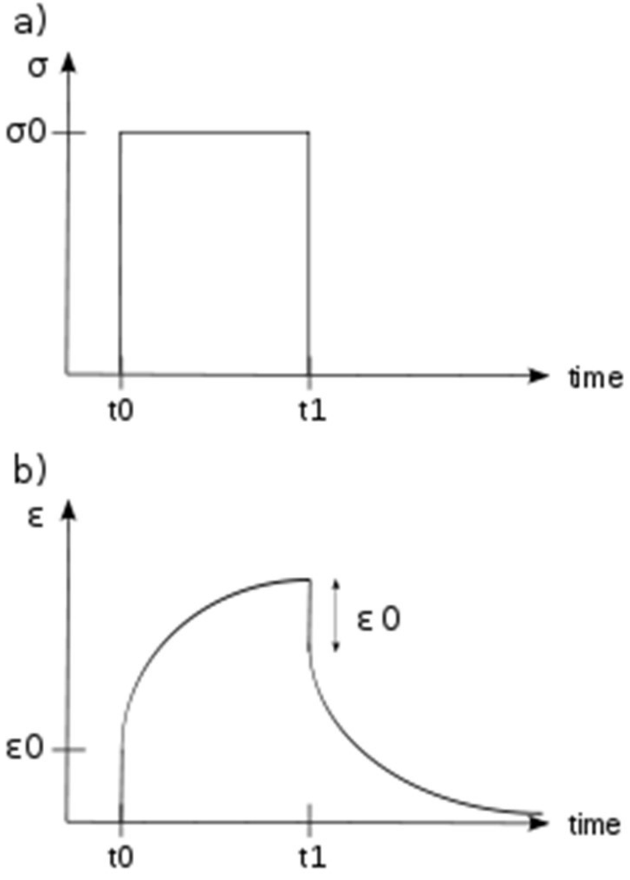
Viscoelasticity

- When a body is suddenly strained and then the strain is maintained constant afterward, the corresponding stresses induced in the body decrease with time. **Relaxation**.
- If the body is suddenly stressed and then the stress is maintained constant afterward, the body continues to deform. **Creep**.
- If the body is subjected to a cyclic loading, the stress-strain relationship in the loading and unloading phases are different. **Hysterisis**.
- Collectively, these phenomena are called features of **viscoelasticity**.

Viscoelasticity

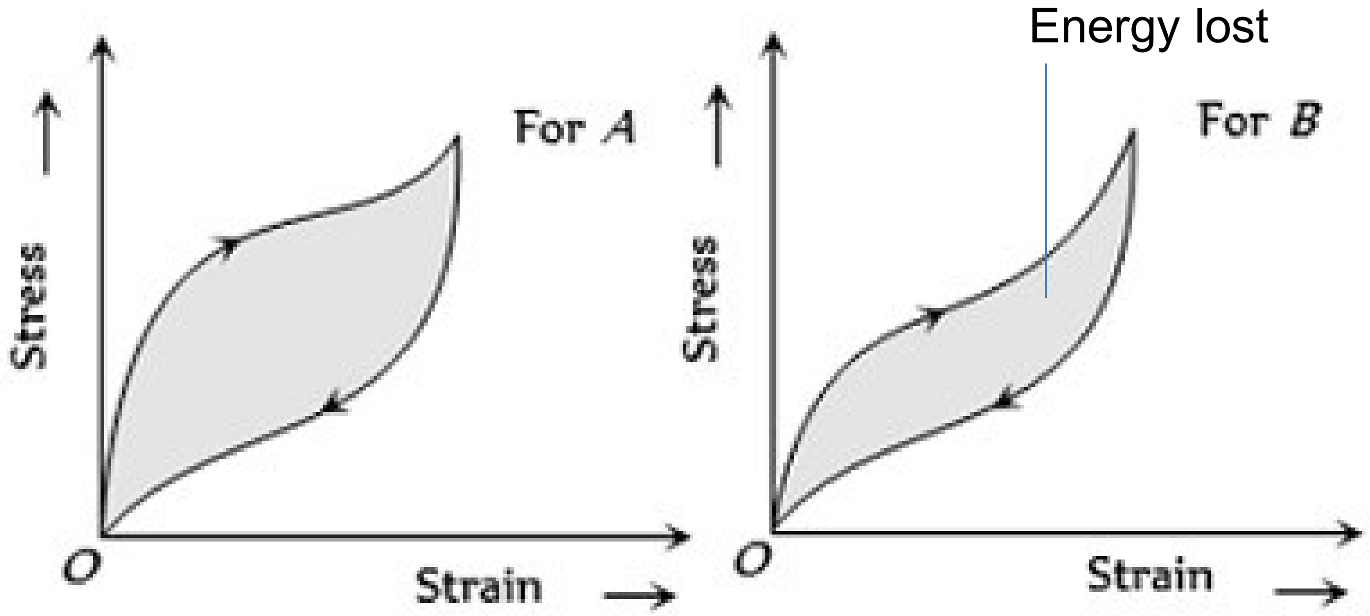


Relaxation



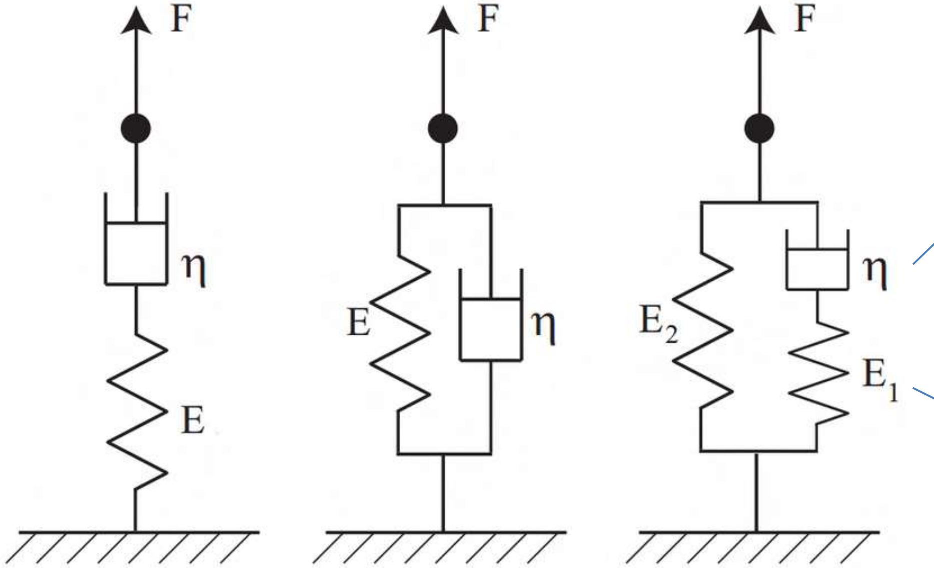
Creep

Viscoelasticity



Hysteresis

Viscoelasticity



Maxwell model

Voigt model

Standard linear model

Dashpot constant

Force

Velocity

dashpot

$$F = \eta \, \dot{u}$$

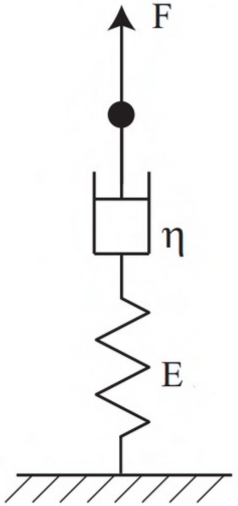
spring

$$F = E \, u$$

Spring constant (modulus of Elasticity)

Displacement

Viscoelasticity



$F = \eta\, \dot{u}$ —— **Maxwell** model

$F = E\, u$

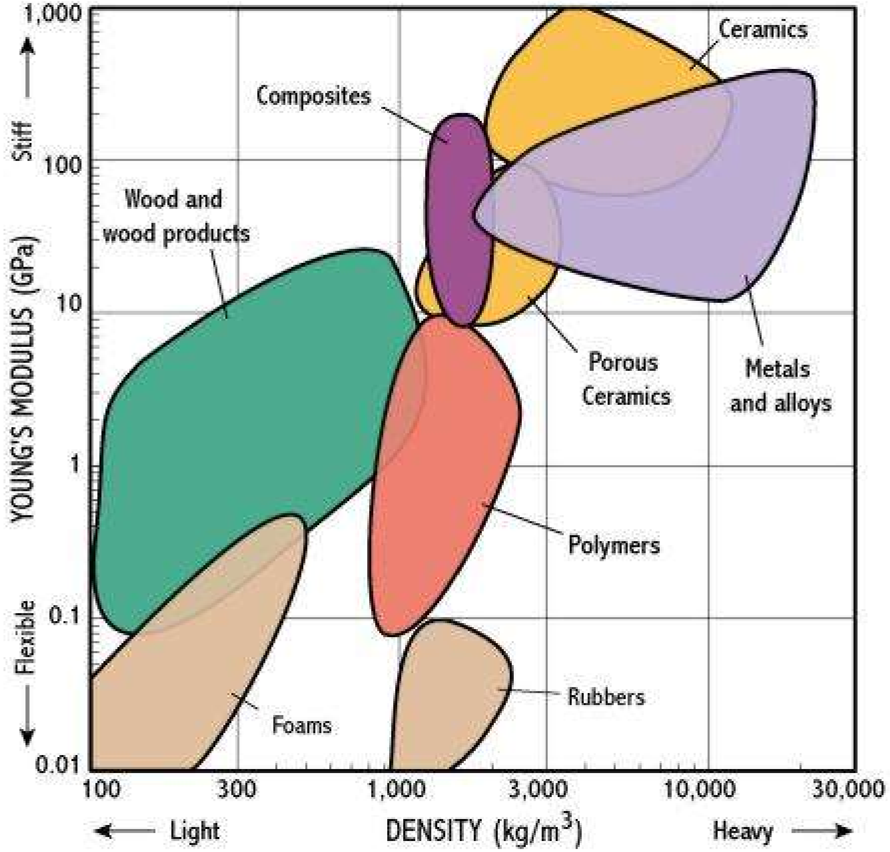
$\dot{u} = \dot{F}/E + F/\eta$

Creep functions

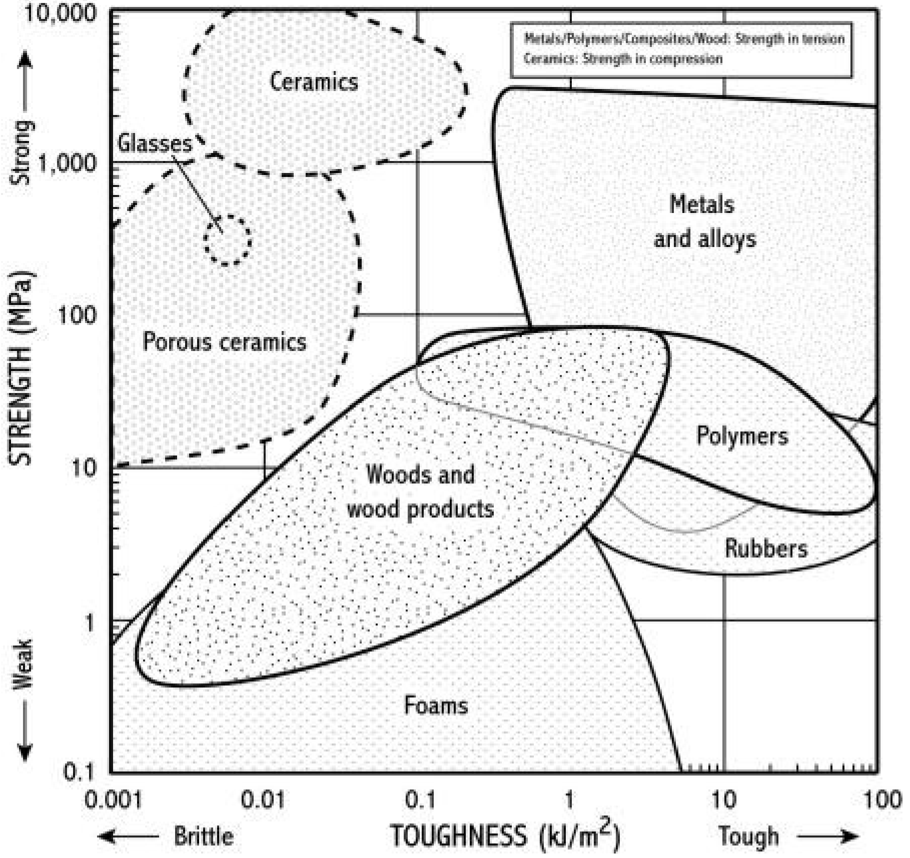
$$c(t) = \left(\frac{1}{E} + \frac{1}{\eta} t\right)$$ **Maxwell** model

$$c(t) = \frac{1}{E}\left(1 - e^{-(E/\eta)t}\right)$$ **Voigt** model

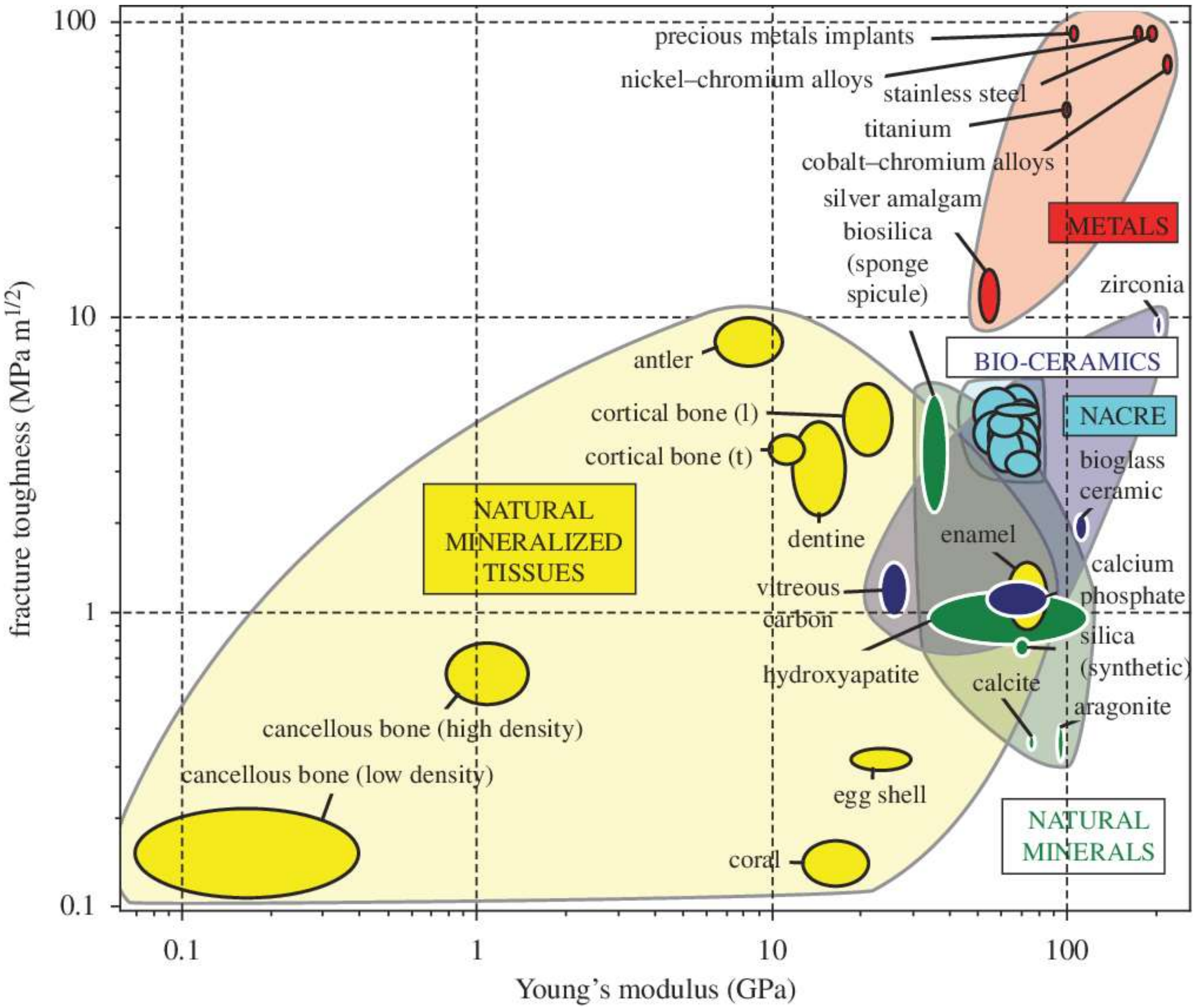
$$c(t) = \frac{1}{E}\left(1 - \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right) e^{-t/\tau_\sigma}\right)$$ **Voigt** model

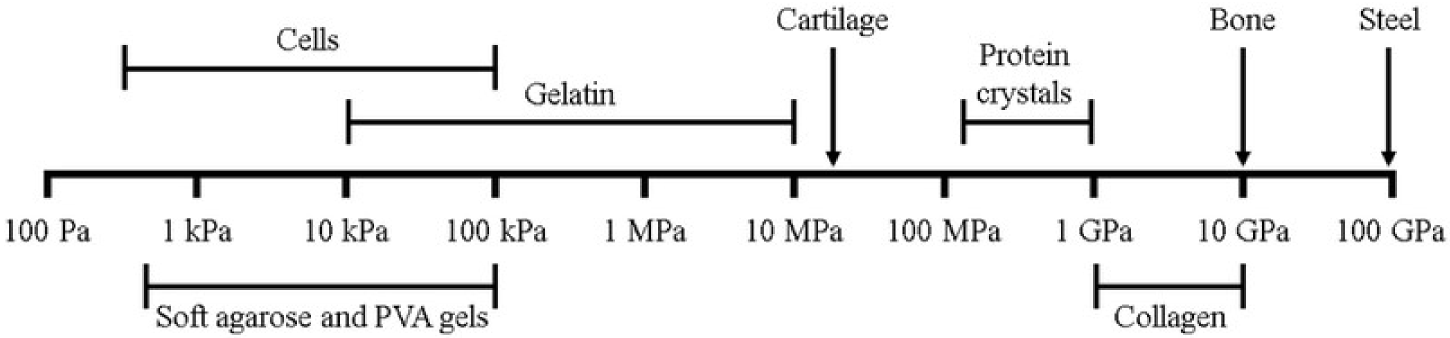


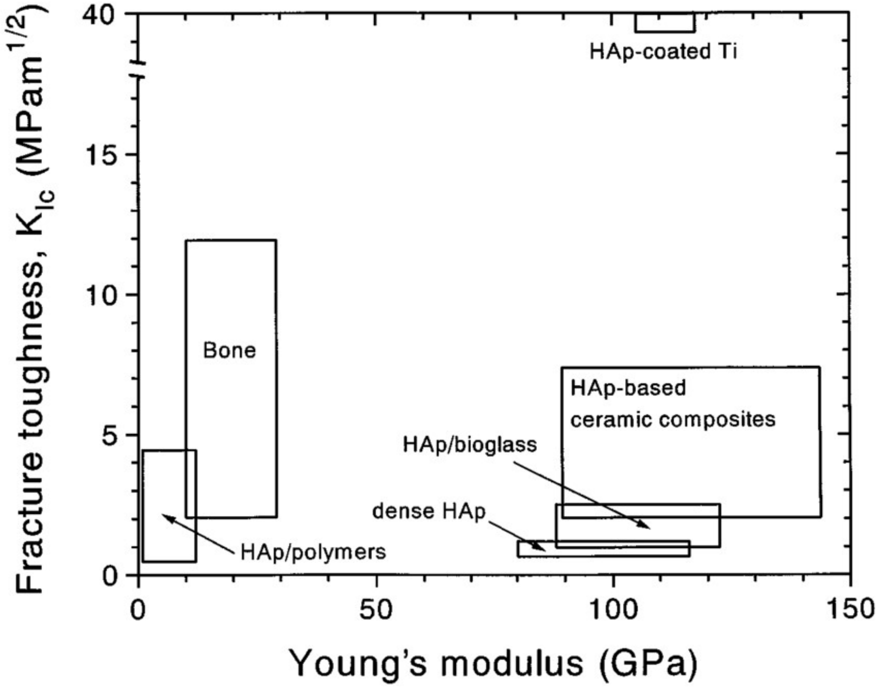
Ceramics
Composites
Wood and wood products
Porous Ceramics
Metals and alloys
Polymers
Foams
Rubbers
YOUNG'S MODULUS (GPa)
Stiff
Flexible
1,000
100
10
1
0.1
0.01
100
300
1,000
3,000
10,000
30,000
Light
DENSITY (kg/m^3)
Heavy



Metals/Polymers/Composites/Wood: Strength in tension
Ceramics: Strength in compression
Ceramics
Glasses
Porous ceramics
Metals and alloys
Polymers
Woods and wood products
Rubbers
Foams
STRENGTH (MPa)
Strong
Weak
10,000
1,000
100
10
1
0.1
0.001
0.01
0.1
1
10
100
Brittle
TOUGHNESS (kJ/m^2)
Tough


100
10
1
0.1
fracture toughness (MPa m$^{1/2}$)
0.1
1
10
100
Young's modulus (GPa)
precious metals implants
nickel–chromium alloys
stainless steel
titanium
cobalt–chromium alloys
silver amalgam
biosilica
(sponge
spicule)
METALS
zirconia
BIO-CERAMICS
NACRE
antler
cortical bone (l)
cortical bone (t)
dentine
bioglass
ceramic
NATURAL
MINERALIZED
TISSUES
enamel
calcium
phosphate
vitreous
carbon
silica
(synthetic)
hydroxyapatite
calcite
aragonite
cancellous bone (high density)
cancellous bone (low density)
egg shell
coral
NATURAL
MINERALS





Material	**Youngs Modulus /GPa**
Mild Steel	210
Copper	120
Bone	18
Plastic	2
Rubber	0.02

Bonds

Potential energy

In physics, potential energy is the energy held by an object because of its position relative to other objects, stresses within itself, its electric charge, or other factors

$$\Delta U = -W$$

← The work done by a conservative force

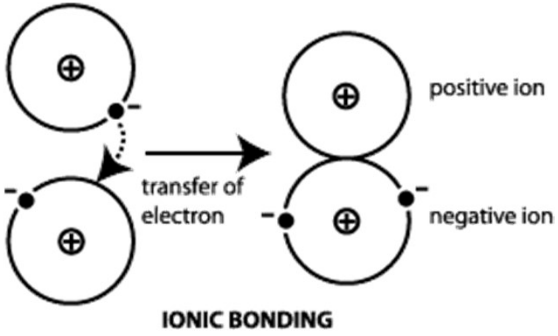
The negative sign provides the convention that work done against a force field increases potential energy, while work done by the force field decreases potential energy.

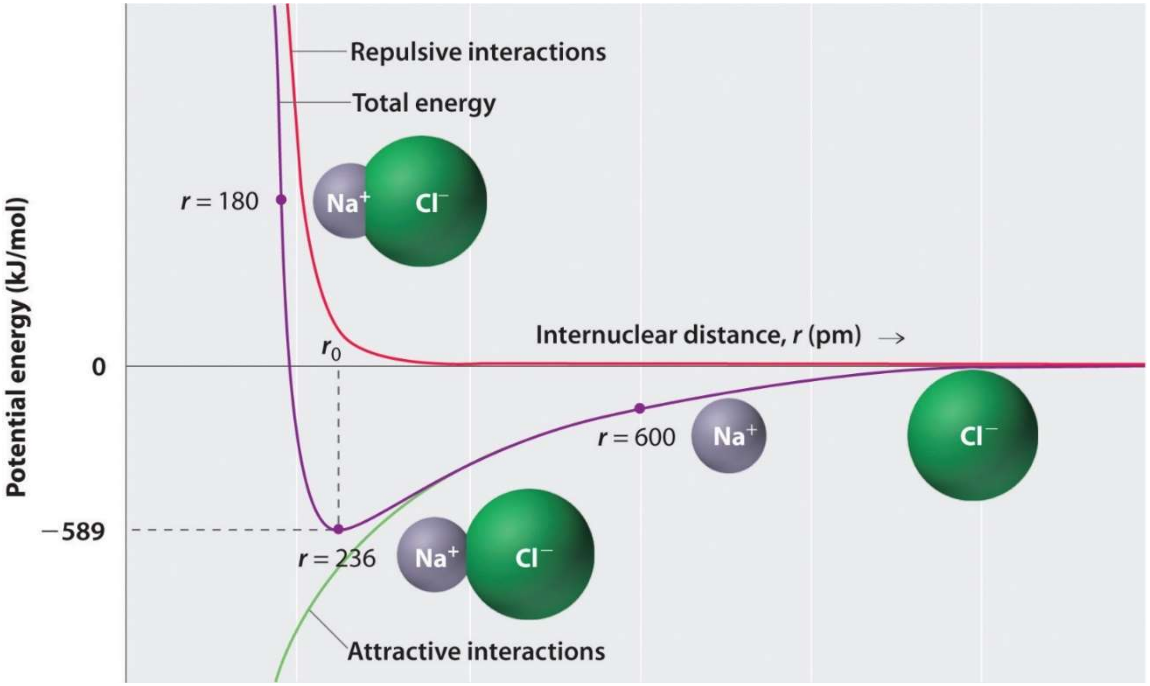
$$\boldsymbol{F} = -\nabla U = -\left(\partial_x, \partial_y, \partial_z\right) U$$

Notably, a force
is derivable from
a potential as…

Ionic bond

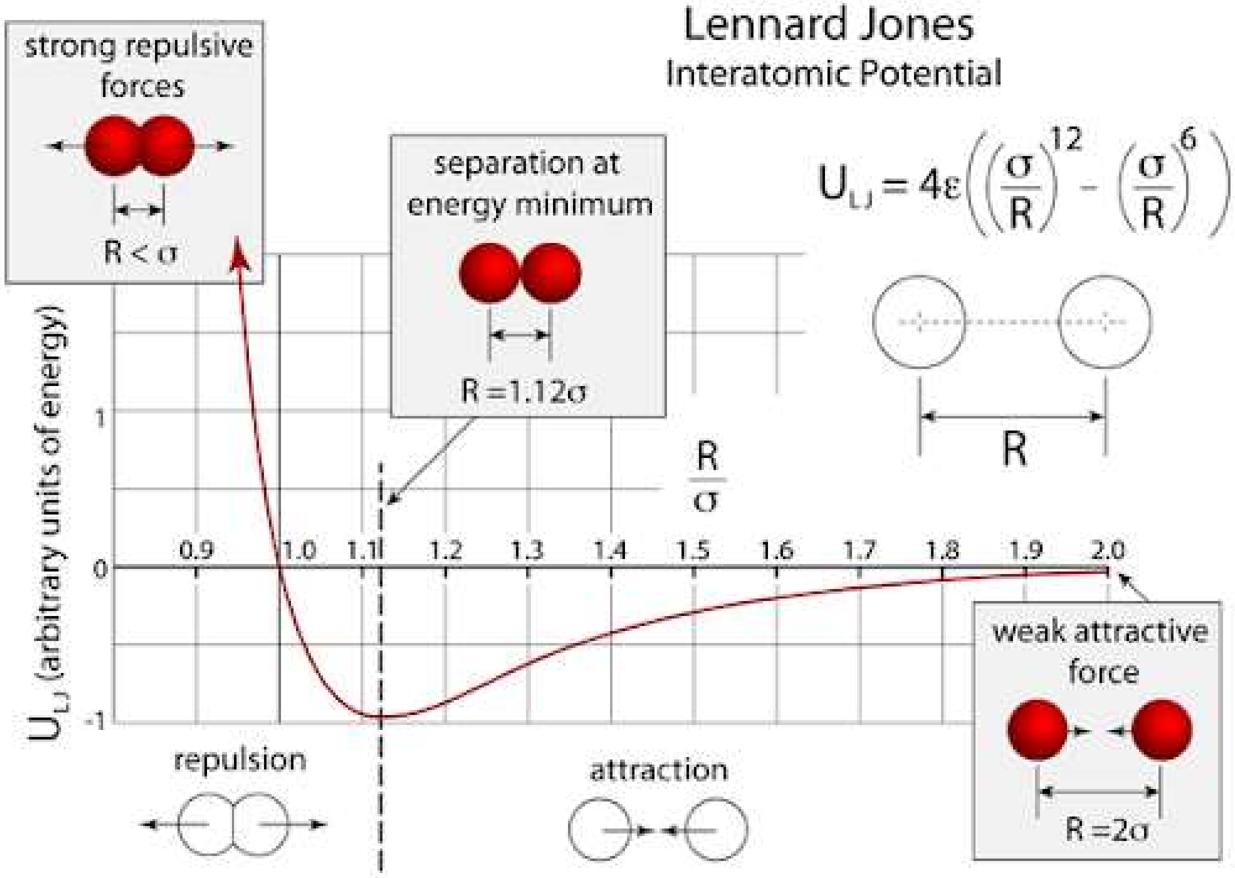
Ions are atoms or molecules which are electrically charged. Ions form when atoms gain or lose electrons. **Ionic bonding** is the attraction between positively- and negatively-charged ions. When many ions attract each other, they form large, ordered, crystal lattices in which each ion is surrounded by ions of the opposite charge. Generally, when metals react with non-metals, electrons are transferred from the metals to the non-metals. In the overall ionic compound, positive and negative charges must be balanced.





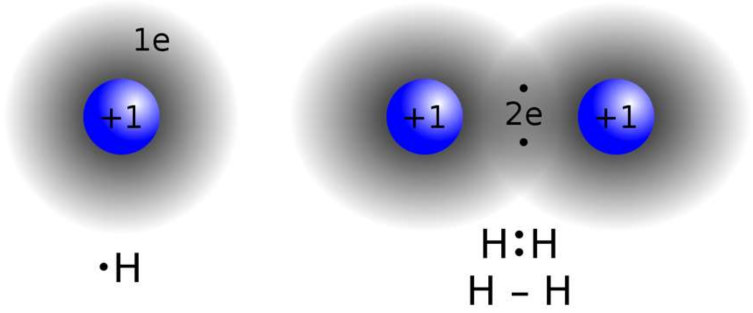
Lennard-Jones potential

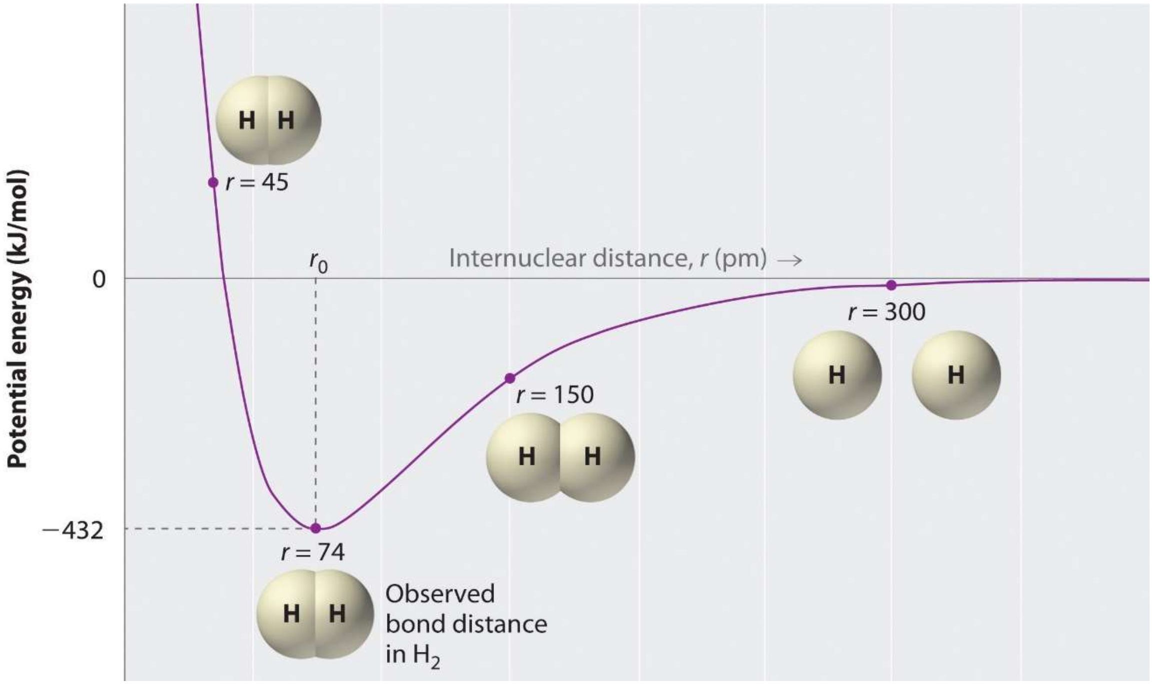
The **Lennard-Jones potential** is a simplified model that yet describes the essential features of interactions between simple atoms and molecules: Two interacting particles repel each other at very close distance, attract each other at moderate distance, and do not interact at infinite distance.



Covalent bond

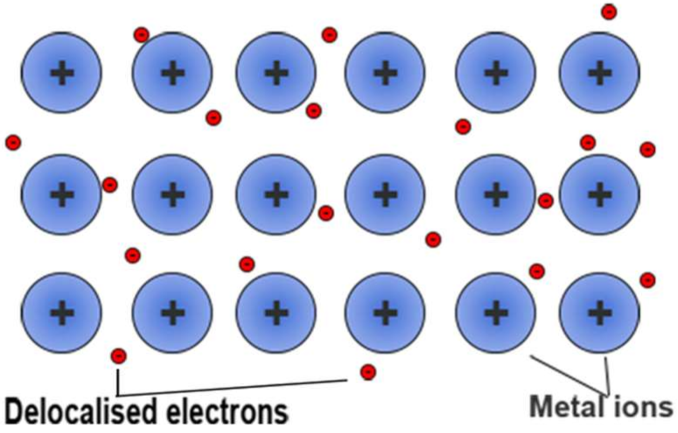
A **covalent bond** is a chemical **bond** that involves the sharing of electron pairs between atoms. These electron pairs are known as shared pairs or **bonding** pairs, and the stable balance of attractive and repulsive forces between atoms, when they share electrons, is known as **covalent bonding**.

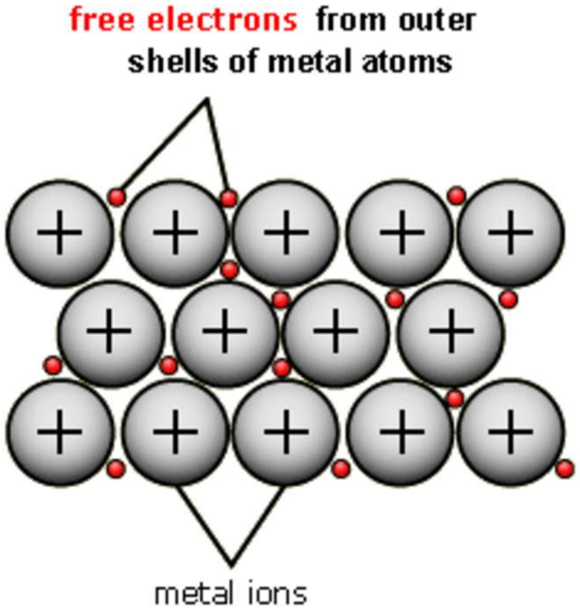




Metallic bond

Metallic bonding is a type of chemical bonding that rises from the electrostatic attractive force between conduction electrons and positively charged metal ions. It may be described as the sharing of free electrons among a structure of positively charged ions.



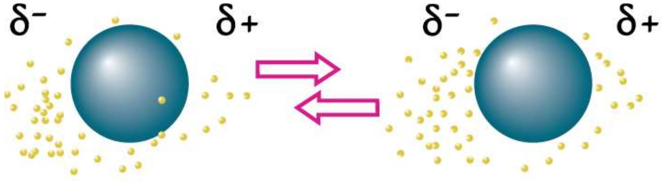


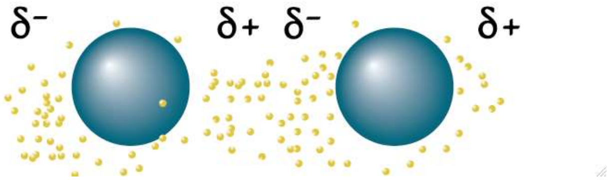
Secondary bonds

Secondary bonds are bonds of a different kind to the primary ones. They are weaker in nature and are broadly classified as Van der Waal's forces and hydrogen bonds. These bonds are due to atomic or molecular dipoles, both permanent and temporary.

Van der Waal's forces are of two types.

The first type is as a result of electrostatic attraction between two permanent dipoles. Permanent dipoles are formed in asymmetric molecules where there are permanent positive and negative regions due to difference in electronegativities of the constituent elements. For example, water molecule is made of one oxygen and two hydrogen atoms



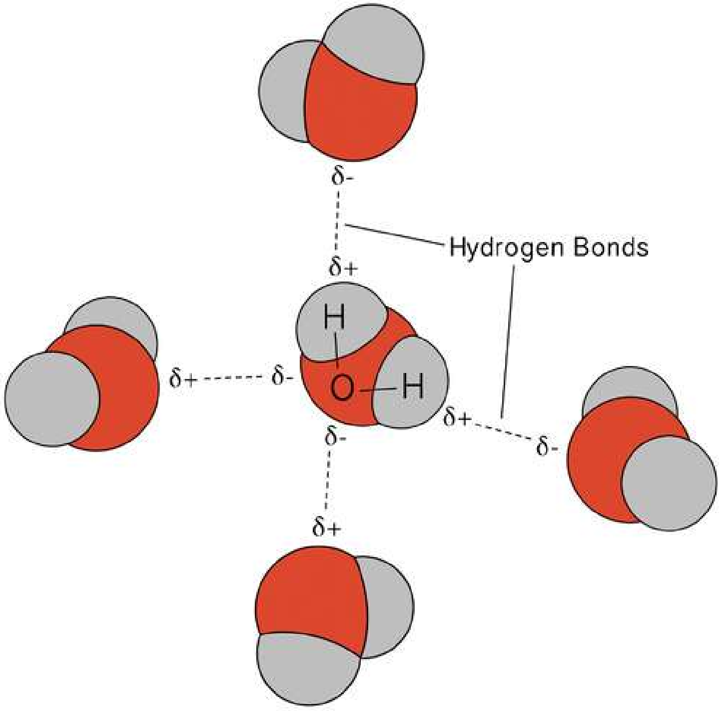


The second type of Van der Waal's bond is formed due to temporary dipoles. A temporary dipole is formed in a symmetric molecule but which has fluctuations of charges giving rise to partial dipole moments for only a few moments.

Secondary bonds

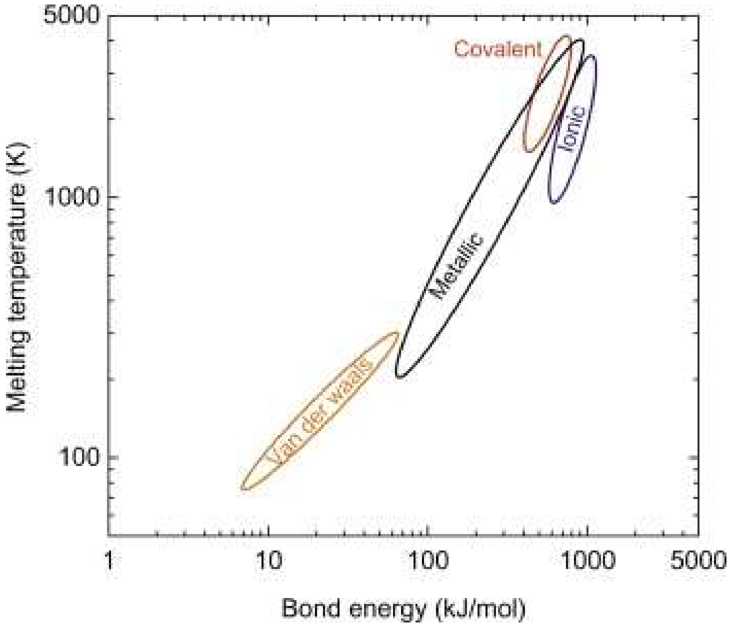
Hydrogen bonds are relatively stronger than Van der Waal's forces but compared to primary bonds they are weak. Bonds between hydrogen atom and atoms of the most electronegative elements (N, O, F) are called hydrogen bonds.

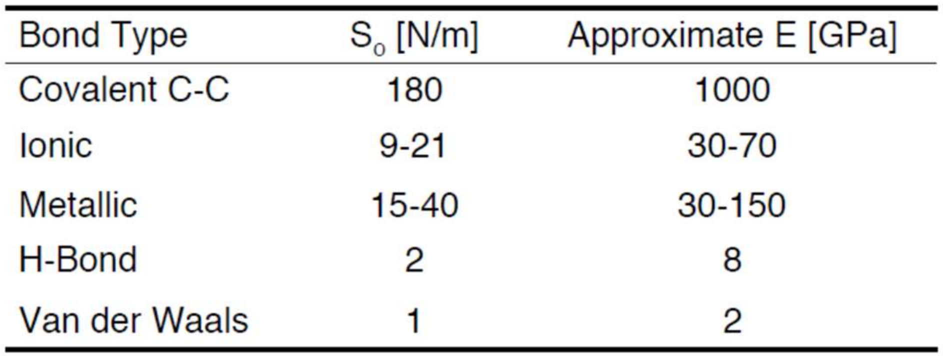
It is based on the fact that hydrogen being the smallest atom provides very little repulsion when interacting with highly electronegative atoms in other molecules and thus succeeds in forming partial bonds with them. This makes hydrogen bonds strong but weaker compared to primary bonds since the interactions here are permanent dipole interactions.



Comparison between different bonds

Type of bond or attraction	Range of bond energies, kJ/mol
Ionic bonds	700-4000
Covalent triple bonds	800-1000
Covalent double bonds	500-700
Covalent single bonds	200-500
Metallic bonds	68-850
Dipole attractions between molecules	40-400
Hydrogen bonds	10-40





Bond Type	S_o [N/m]	Approximate E [GPa]
Covalent C-C	180	1000
Ionic	9-21	30-70
Metallic	15-40	30-150
H-Bond	2	8
Van der Waals	1	2